Computation of Koszul homology and application to partial differential systems

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Consider a PDEs system

$$(\Sigma):\ F^{\rho}\left(x^i, \frac{\partial^{|\mu|}u^j}{\partial x^{\mu}}\right) = 0$$

➔ $x^1, \cdots, x^n$, $u^1, \cdots, u^p$: (in)dependent variables

➔ $F^1, \cdots, F^q$: equations of the system

Formal theory

Geometric methods

➔ integral manifolds

➔ differential forms

➔ jet bundles

Algebraic methods

➔ differential elimination

➔ integrability conditions

➔ homological algebra

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Geometric methods

- ➔ integral manifolds
- ➔ differential forms
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Algebraic methods

- ➔ **differential elimination**
- ➔ **integrability conditions**
- ➔ **homological algebra**

A fundamental notion: **formal integrability**

➔ derivatives of F^ρ's do not bring new integrability conditions

Examples

$(\Sigma_1): \; \dot{u} \; = \; u$

➔ (Σ_1) is formally integrable with formal power series solutions

$$u_0 + u_0 t + \frac{u_0}{2} t^2 + \frac{u_0}{3!} t^3 + \dots$$

$(\Sigma_2): \; u_{33} = u_{13}, \quad u_{23} = u_{13}, \quad u_{12} = u_{11}$

➔ derivatives of (Σ_2) yield coherent results, *e.g.*,

$$(u_{33})_2 \; = \; u_{123} \; = \; u_{113} \qquad \text{and} \qquad (u_{23})_3 \; = \; u_{133} \; = \; u_{113}$$

➔ (Σ_2) is formally integrable with formal power series solutions

$$u_0 + u_i x^i + \frac{u_{11}}{2}(x^1)^2 + u_{11} x^1 x^2 + u_{13} x^1 x^3 + \frac{u_{22}}{2}(x^2)^2 + u_{13} x^2 x^3 + \frac{u_{13}}{2}(x^3)^2 + \dots$$

Integrability defect and homology

Janet example. $(\Sigma): \; u_{33} = x^2 u_{11}, \quad u_{22} = 0$

➔ $(u_{22})_{33} = 0$ and

$$(u_{33})_{22} \;=\; \frac{\partial}{\partial x^2}\left(x^2 u_{112} + u_{11}\right) \;=\; x^2 u_{1122} + 2u_{112} \;=\; 2u_{112}$$

➔ **new integrability condition** $u_{112} = 0$

Remark. The **Spencer cohomology does not vanish in homological degree** 2

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Remark. The Spencer cohomology does not vanish in homological degree 2

Theorem (Goldschmidt). If **the symbol of Σ is 2-acyclic** and if 1st derivatives of Σ do not bring new integrability conditions, then Σ is formally integrable

Theorem (Serre). Spencer and Koszul complexes are dual to each other

➔ 2-acyclicity ⇔ **Koszul homology** vanishes in degree 2

Our contribution:

effectively check 2-acyclicity of **linear systems**

➔ OreMorphisms and OreModules **Maple** packages

Let a system $(\Sigma): \ \forall 1 \leq i \leq q, \quad R^i_1 u^1 + \cdots + R^i_p u^p = 0$

➔ described by $\left(R^i_j\right) \in \mathcal{D}^{q \times p}$ ($\mathcal{D}$: ring of partial differential operators)

Symbol module: $\mathcal{M}(\Sigma) := \mathcal{A}(0)^{1\times p} / \left(\mathcal{A}(-d)^{1\times q}\sigma(R)\right)$ (d: order of Σ)

➔ $\mathcal{A} := \mathrm{gr}(\mathcal{D})$ and $\sigma(R)$: **top-order part** of R

Koszul homology: homology $H_k \simeq \ker(\partial_k) / \operatorname{im}(\partial_{k+1})$ of the **Koszul complex**

$$0 \to \Lambda^n T \otimes \mathcal{M}(\Sigma) \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_3} \Lambda^2 T \otimes \mathcal{M}(\Sigma) \xrightarrow{\partial_2} T \otimes \mathcal{M}(\Sigma) \xrightarrow{\partial_1} \mathcal{M}(\Sigma) \to 0$$

where for basis elements (χ_i) of $T = \mathcal{A}^n_1$

$$\partial_k\left(\chi_1 \wedge \cdots \wedge \chi_k \otimes m\right) := \sum_{i=1}^{k} (-1)^{i+1} \chi_1 \wedge \cdots \wedge \widehat{\chi_i} \wedge \cdots \wedge \chi_k \otimes \chi_i m$$

Definitions. $\mathcal{M}(\Sigma)$ is 2-acyclic if H_2 vanishes, and involutive if $H_2, \cdots, H_n$ vanish

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Objective: compute H_2 ➔ using `OreMorphisms` package

Approach: given $\mathcal{M}_2 \xrightarrow{f} \mathcal{M}_1 \xrightarrow{g} \mathcal{M}_0$ s.t. $g \circ f = 0$ and $\mathcal{M}_i = \mathcal{A}^{1\times p_i} / \left(\mathcal{A}^{1\times q_i} R_i\right)$

$$\begin{array}{ccccccc}
\mathcal{A}^{1\times q_2} & \xrightarrow{.R_2} & \mathcal{A}^{1\times p_2} & \longrightarrow & \mathcal{M}_2 & \longrightarrow & 0 \\
\downarrow{\scriptstyle .Q_2} & & \downarrow{\scriptstyle .P_2} & & \downarrow{\scriptstyle f} & & \\
\mathcal{A}^{1\times q_1} & \xrightarrow{.R_1} & \mathcal{A}^{1\times p_1} & \longrightarrow & \mathcal{M}_1 & \longrightarrow & 0 \\
\downarrow{\scriptstyle .Q_1} & & \downarrow{\scriptstyle .P_1} & & \downarrow{\scriptstyle g} & & \\
\mathcal{A}^{1\times q_0} & \xrightarrow{.R_0} & \mathcal{A}^{1\times p_0} & \longrightarrow & \mathcal{M}_0 & \longrightarrow & 0
\end{array}$$

`OreMorphisms` computes matrices S', S'' s.t. $\ker \begin{pmatrix} .P_1 \\ .R_0 \end{pmatrix} = \mathcal{A}^{1\times s'} \begin{pmatrix} S' & -S'' \end{pmatrix}$ and

$$\ker(g) / \operatorname{im}(f) = \mathcal{A}^{1\times s'} S' / \left(\mathcal{A}^{1\times (p_2+q_2)} \begin{pmatrix} P_2 \\ R_1 \end{pmatrix}\right)$$

Case of H_2 Koszul homology: $\mathcal{M}_i = \Lambda^i T \otimes \mathcal{M}(\Sigma)$, $f = \partial_{i+1}, g = \partial_i$

➔ $R_i = I \otimes \sigma(R)$; $P_i, Q_i = (\text{grad}, \text{curl}, \text{div}) \otimes I$

Theorem (C., Cluzeau, Quadrat). Let (Σ) be a linear system of PDEs and let $\mathcal{M}(\Sigma)$ be the symbol module of (Σ). Let S, S' be two matrices such that

$$H_2 \;=\; \mathcal{A}^{1\times s'} S' / \left(\mathcal{A}^{1\times s} S\right)$$

Then, $\mathcal{M}(\Sigma)$ is 2-acyclic iff the top degree part of S' is a left-multiple of S.

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$$H_2 = \mathcal{A}^{1\times s'} S' / \left(\mathcal{A}^{1\times s} S\right)$$

Then, $\mathcal{M}(\Sigma)$ is 2-acyclic iff the top degree part of S' is a left-multiple of S.

Algorithmic side of the theorem

Existence of a right factorization may be checked using `OreModules`

$(\Sigma): \ u_{33} = u_{13}, \ u_{23} = u_{13}, \ u_{12} = u_{11}$

Symbol module: $\mathcal{A} := \mathbb{Q}[\chi_1, \chi_2, \chi_3]$

$$\mathcal{M}(\Sigma) \ = \ \mathcal{A}/\left(\mathcal{A}(-2)^{1\times 3}\begin{pmatrix} \chi_3^2 - \chi_1\chi_3 \\ \chi_2\chi_3 - \chi_1\chi_3 \\ \chi_1\chi_2 - \chi_1^2 \end{pmatrix}\right)$$

2nd homology group: $H_2 = \mathcal{A}^{1\times 7} S' / \left(\mathcal{A}^{1\times 10} S\right)$

$$S' = \begin{pmatrix} \chi_3 & \chi_3 & \chi_3 \\ \chi_1 & \chi_2 & \chi_3 \\ 0 & \chi_1 - \chi_2 & 0 \\ 0 & \chi_2\,\chi_3 - \chi_3^2 & 0 \\ 0 & 0 & \chi_2\,\chi_3 - \chi_3^2 \\ 0 & 0 & \chi_1\,\chi_3 - \chi_3^2 \\ 0 & 0 & \chi_1^2 - \chi_1\,\chi_2 \end{pmatrix} \qquad S = \begin{pmatrix} \chi_1 & \chi_2 & \chi_3 \\ \chi_3^2 - \chi_1\,\chi_3 & 0 & 0 \\ \chi_2\,\chi_3 - \chi_1\,\chi_3 & 0 & 0 \\ \chi_1\,\chi_2 - \chi_1^2 & 0 & 0 \\ 0 & \chi_3^2 - \chi_1\,\chi_3 & 0 \\ 0 & \chi_2\,\chi_3 - \chi_1\,\chi_3 & 0 \\ 0 & \chi_1\,\chi_2 - \chi_1^2 & 0 \\ 0 & 0 & \chi_3^2 - \chi_1\,\chi_3 \\ 0 & 0 & \chi_2\,\chi_3 - \chi_1\,\chi_3 \\ 0 & 0 & \chi_1\,\chi_2 - \chi_1^2 \end{pmatrix}$$

2-acyclicity test: **success**

(Σ) : $u_{33} = u_{22} = u_{13} = u_{12} = 0,\ u_{23} = \alpha u_{11}$ (α: a parameter)

Symbol module: $\mathcal{A} := \mathbb{Q}[\alpha][\chi_1, \chi_2, \chi_3]$

$$\mathcal{M}_\alpha(\Sigma) = \mathcal{A}/\left(\mathcal{A}(-2)^{1\times 5}\begin{pmatrix}\chi_3^2 & \chi_2\chi_3 - \alpha\chi_1^2 & \chi_2^2 & \chi_1\chi_3 & \chi_1\chi_2\end{pmatrix}^T\right)$$

2nd homology group: $H_2 = \mathcal{A}^{1\times 11} S' / \left(\mathcal{A}^{1\times 16} S\right)$

$$S' = \begin{pmatrix} \chi_2 & 0 & \alpha\,\chi_1 \\ \chi_3 & \alpha\,\chi_1 & 0 \\ \chi_1 & 0 & 0 \\ \chi_3^2 & 0 & 0 \\ \chi_2\,\chi_3 & 0 & 0 \\ \chi_2^2 & 0 & 0 \\ 0 & \chi_3 & 0 \\ 0 & \chi_2 & \chi_3 \\ 0 & 0 & \chi_2 \\ 0 & 0 & \chi_3^2 \\ 0 & 0 & \chi_1\,\chi_3 \end{pmatrix} \qquad S = \begin{pmatrix} \chi_1 & \chi_2 & \chi_3 \\ \chi_3^2 & 0 & 0 \\ \chi_2\,\chi_3 - \alpha\,\chi_1^2 & 0 & 0 \\ \chi_2^2 & 0 & 0 \\ \chi_1\,\chi_3 & 0 & 0 \\ \chi_1\,\chi_2 & 0 & 0 \\ 0 & \chi_3^2 & 0 \\ 0 & \chi_2\,\chi_3 - \alpha\,\chi_1^2 & 0 \\ 0 & \chi_2^2 & 0 \\ 0 & \chi_1\,\chi_3 & 0 \\ 0 & \chi_1\,\chi_2 & 0 \\ 0 & 0 & \chi_3^2 \\ 0 & 0 & \chi_2\,\chi_3 - \alpha\,\chi_1^2 \\ 0 & 0 & \chi_2^2 \\ 0 & 0 & \chi_1\,\chi_3 \\ 0 & 0 & \chi_1\,\chi_2 \end{pmatrix}$$

2-acyclicity test: **success**. Moreover, $\mathcal{M}_\alpha(\Sigma)$ is **involutive** iff $\alpha \neq 0$

$(\Sigma):\ \ u_{33} = x^2 u_{11},\ \ u_{22} = 0$

Symbol module: $\mathcal{A} := \mathbb{Q}[x^1, x^2, x^3][\chi_1, \chi_2, \chi_3]$

$$\mathcal{M}_\alpha(\Sigma) \ = \ \mathcal{A}/\left(\mathcal{A}(-2)^{1\times 2}\begin{pmatrix}\chi_3^2 - x^2\chi_1^2 \\ \chi_2^2\end{pmatrix}\right)$$

2nd homology group: $H_2 = \mathcal{A}^{1\times 8} S' / \left(\mathcal{A}^{1\times 7} S\right)$

$$S' = \begin{pmatrix} \chi_1 & \chi_2 & \chi_3 \\ \chi_2^2 & 0 & 0 \\ \chi_2\,\chi_3 & 0 & \chi_1\,\chi_2 \\ \chi_3^2 & x_2\,\chi_1\,\chi_2 & x_2\,\chi_1\,\chi_3 \\ 0 & \chi_2^2 & 0 \\ 0 & x_2\,\chi_1^2 - \chi_3^2 & 0 \\ 0 & 0 & \chi_2^2 \\ 0 & 0 & x_2\,\chi_1^2 - \chi_3^2 \end{pmatrix} \qquad S = \begin{pmatrix} \chi_1 & \chi_2 & \chi_3 \\ \chi_3^2 - x_2\,\chi_1^2 & 0 & 0 \\ \chi_2^2 & 0 & 0 \\ 0 & \chi_3^2 - x_2\,\chi_1^2 & 0 \\ 0 & \chi_2^2 & 0 \\ 0 & 0 & \chi_3^2 - x_2\,\chi_1^2 \\ 0 & 0 & \chi_2^2 \end{pmatrix}$$

2-acyclicity test: **fail**

Conclusion and perspectives

Summary of presented results

- ➔ new criterion for 2-ayclicity of linear PDEs systems
- ➔ effective test using `OreMorphisms` and `OreModules` packages
- ➔ illustration with various classes of linear systems

Remark. More generally, involutivity can be checked with `OreMorphism` and `OreModules`

Further works

- ➔ go further in the effective approach to Spencer cohomology
  (Koszul-Tate theory, Spencer sequences)
- ➔ applications to physics and control theory
  (elasticity theory, hydrodynamics, electromagnetism, general relativity)

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THANK YOU FOR LISTENING!