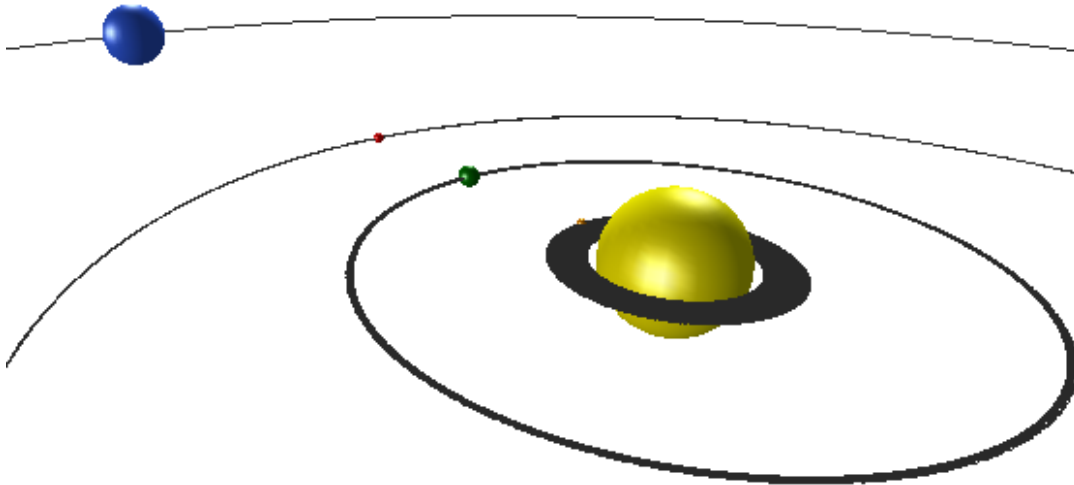


Orbital Mechanics



The objects that orbit earth have only a few forces acting on them, the largest being the gravitational pull from the earth. The trajectories that satellites or rockets follow are largely determined by the central force of gravity. Johannes Kepler developed the laws of planetary motion we use today to predict the motion of the planets about the sun or the path of a satellite about the earth, and his theories were confirmed when Newton revealed his universal law of gravitation. These laws provide a good approximation of the path of a body in space mechanics.

This module explains angular momentum, central forces including Newton's law of gravitation, properties of elliptic orbits such as eccentricity, and Kepler's laws of planetary motion. The associated equations are presented alongside an example in MapleSim and an interactive plot of the eccentricity of an orbit. It is recommended that the Rotational Kinematics module is reviewed prior to completing this module.

Angular Momentum

Consider an object moving with linear momentum as described by $m \cdot v$ in the Impulse and Momentum module. The moment of this object about point O is the moment of momentum, or **angular momentum**.

$$\vec{H}_O = \vec{r} \times m \cdot \vec{v}$$

... Eq. (1)

The cross product implies that the angular momentum is a vector perpendicular to both the momentum $m \cdot \vec{v}$ and the position \vec{r} vectors. The magnitude of the angular momentum vector is given as

$$H_O = r \cdot m \cdot v \cdot \sin(\phi) = r \cdot m \cdot v_\theta$$

... Eq. (2)

Where phi ϕ is the angle between the position vector and momentum vector, and v_θ is the tangential speed in a rotating frame of reference. Substituting the tangential velocity, this can be written in the form

$$H_O = m \cdot r^2 \cdot \omega$$

... Eq. (3)

where ω is the angular speed. Deriving the angular momentum in vector form gives

$$\frac{d}{dt} \vec{H}_O = \vec{v} \times m \cdot \vec{v} + \vec{r} \times m \cdot \vec{a}$$

... Eq. (4)

The cross product of the first term will be zero. The product $m \cdot a$ is a force, which is crossed with the position, resulting in a moment. The rate of change of angular momentum becomes

$$\sum \vec{M}_O = \frac{d}{dt} \vec{H}_O$$

... Eq. (5)

It is often more advantageous to express angular momentum per unit mass, denoted by h .

$$h = \frac{H_O}{m} = r^2 \cdot \omega$$

... Eq. (6)

Central Forces

If an object is undergoing a force directed toward a fixed point and no other forces are acting on this object, it is experiencing a central force. Gravity is a common central force. A satellite in orbit experiencing no other (or negligible) forces except for the gravitational pull towards the center of mass of the earth is undergoing a central force. Therefore, if the force on the object is directed at the center of mass, there is no moment on the object.

$$\sum \vec{M} = 0 = \frac{d}{dt} \vec{H}_O$$

... Eq. (7)

This must mean that the angular momentum is a constant value for objects under a central force.

Newton's Law of Universal Gravitation

Newton states his law of gravitation as

$$F = \frac{G \cdot M \cdot m}{r^2}$$

... Eq. (8)

where the force F is related to the product of mass $M \cdot m$ of the attracting objects, the distance r between them, multiplied by the constant of gravitation $G = 66.73 \times 10^{-12} \text{ m}^3 / \text{kg} \cdot \text{s}^2$.

The weight that all objects experience under gravity on the surface of a planet of radius R is

$$W = m \cdot g = \frac{G \cdot M \cdot m}{R^2}$$

... Eq. (9)

Trajectory of Particles Under a Central Force

Particles moving under a central force follow the trajectory given by the following differential equation.

$$\frac{d^2 u}{d\theta^2} + u = \frac{F}{m \cdot h^2 \cdot u^2}$$

... Eq. (10)

Where the force F is assumed as an attractive force (positive towards the origin), and u represents $1/r$.

Deriving the Trajectory Differential Equation

The radial and transverse components describe the motion of a particle in terms of r and θ .

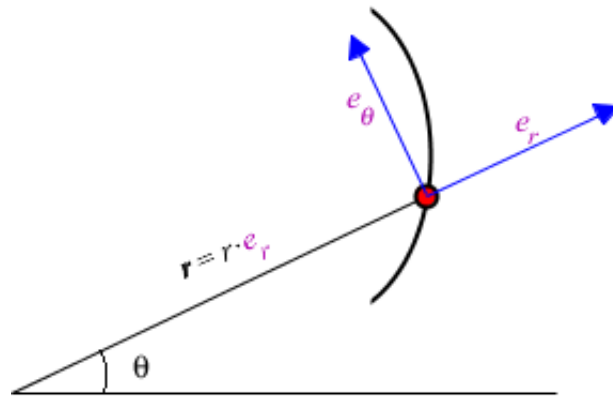


Figure 1: Radial components of a trajectory

The velocity of the particle is the time derivative of its position.

$$\mathbf{v} = \frac{d}{dt} (r \cdot \mathbf{e}_r) = \dot{r} \cdot \mathbf{e}_r + r \cdot \dot{\mathbf{e}}_r$$

... Eq. (11)

The derivative of the unit vector \mathbf{e}_r can be represented with \mathbf{e}_θ .

$$\dot{\mathbf{e}}_r = \dot{\theta} \cdot \mathbf{e}_\theta$$

$$\dot{\mathbf{e}}_\theta = -\dot{\theta} \cdot \mathbf{e}_r$$

... Eq. (12a, 12b)

Substituting this in provides the velocity of a particle in radial and transverse components

$$\mathbf{v} = \dot{r} \cdot \mathbf{e}_r + r \cdot \dot{\theta} \cdot \mathbf{e}_\theta$$

... Eq. (13)

Following the same method, differentiating with respect to time and simplifying provides the acceleration of the particle.

$$\mathbf{a} = (\ddot{r} - r \cdot \dot{\theta}^2) \mathbf{e}_r + (r \cdot \ddot{\theta} + 2 \cdot \dot{r} \cdot \dot{\theta}) \mathbf{e}_\theta$$

... Eq. (14)

The force applied to the particle given in terms of the radial and transverse components follows $F=ma$.

$$\sum F_r = m \cdot a_r = m \cdot (\ddot{r} - r \cdot \dot{\theta}^2)$$

$$\sum F_{\theta} = m \cdot a_{\theta} = m \cdot (r \cdot \ddot{\theta} + 2 \cdot \dot{r} \cdot \dot{\theta})$$

... Eq. (15a, 15b)

Under a central force, the force experience by a particle will be along the frame directed towards the origin.

$$\sum F = -F \cdot e_r$$

... Eq. (16)

Including the force in eq. (15) resolves the equations to

$$m \cdot (\ddot{r} - r \cdot \dot{\theta}^2) = -F$$

$$m \cdot (r \cdot \ddot{\theta} + 2 \cdot \dot{r} \cdot \dot{\theta}) = 0$$

... Eq. (17a, 17b)

From eq. (6),

$$\omega = \dot{\theta} = \frac{h}{r^2}$$

... Eq. (18)

Similarly, velocity can be expressed with angular momentum.

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = \frac{dr}{d\theta} \cdot \left(\frac{h}{r^2} \right) = -h \cdot \frac{d}{d\theta} \left(\frac{1}{r} \right)$$

... Eq. (19)

Acceleration is found in a similar fasion.

$$\ddot{r} = \frac{h}{r^2} \cdot \frac{d\dot{r}}{d\theta} = \frac{h}{r^2} \cdot \frac{d}{d\theta} \left[-h \cdot \frac{d}{d\theta} \left(\frac{1}{r} \right) \right] = -\frac{h^2}{r^2} \cdot \frac{d^2}{d\theta^2} \left(\frac{1}{r} \right)$$

... Eq. (20)

Let 1/r be represented with u. Substituting _ and _ into eq. (17a) shows

$$m \cdot \left(-h^2 \cdot u^2 \cdot \frac{d^2}{d\theta^2} (u) - \frac{h^2 \cdot u^4}{u} \right) = -F$$

... Eq. (21)

Simplifying this resolves to a differential equation.

$$\frac{d^2 u}{d\theta^2} + u = \frac{F}{m \cdot h^2 \cdot u^2}$$

... Eq. (10)

If the central force is gravity, using Newton's equation from eq. (8) the differential equation represents the trajectory of an object in orbit.

$$\frac{d^2u}{d\theta^2} + u = \frac{G \cdot M}{h^2}$$

... Eq. (22)

Solve the ODE by adding the particular solution $u=GM/h^2$ to the general solution $u=C \cdot \cos(\theta - \theta_0)$ with $\theta_0=0$.

$$\frac{1}{r} = u = \frac{G \cdot M}{h^2} + C \cdot \cos(\theta)$$

... Eq. (23)

The ratio of the constants describe the **eccentricity** of the trajectory.

$$\varepsilon = \frac{C \cdot h^2}{G \cdot M}$$

... Eq. (24)

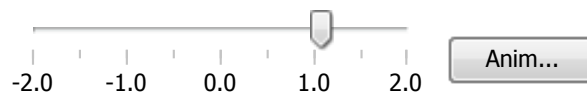
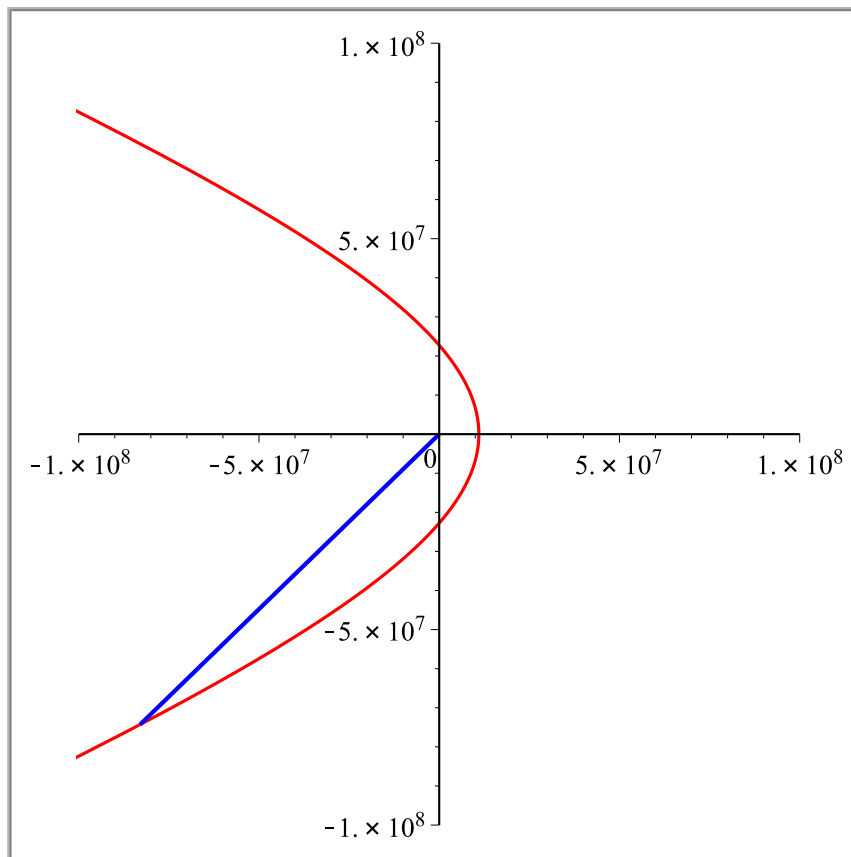
This allows eq. (23) to be written in the following form.

$$\frac{1}{r} = \frac{G \cdot M}{h^2} (1 + \varepsilon \cdot \cos(\theta))$$

... Eq. (25)

Exploring Eccentricity

Observe the effects of vaying the eccentricity of the particles trajectory in the interactive section below.



Eccentricity = 1.07 , Path = Hyperbola

Figure 2: Visualizing the eccentricity of orbits

The eccentricity of an orbit describes the path of the particle quite well. There are four cases of trajectories that concern the eccentricity.

1. $\epsilon > 1$. There are two angles which evaluate the right hand side of eq. (25) to zero, therefore the vector r tends to infinity. The particle will travel in a **hyperbolic** trajectory.
2. $\epsilon = 1$. A **parabolic** path is followed. The radius vector is infinite at $\theta = 180^\circ$.
3. $0 < \epsilon < 1$. The conic section becomes an **ellipse** as there is a radius vector for every value of θ .
4. $\epsilon = 0$. The radius of the ellipse stays constant and therefore describes a **circular** orbit.

A spacecraft launched into orbit begins its orbit at $\theta = 0$, shown in the figure below.

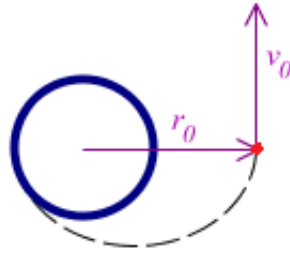


Figure 3: Initial position and velocity at $\theta = 0$.

At this point, it has an initial position r_0 and velocity v_0 and the object is located at the perigee, the closest point to earth in its orbit. The constant C can be found with r_0 and v_0 for any orbit. At $\theta = 0$, eq. (25) becomes

$$\frac{1}{r_0} = \frac{G \cdot M}{h^2} \cdot (1 + \varepsilon \cdot \cos(0))$$

... Eq. (26)

Substituting the eccentricity ε and expanding the equation reveals the constant C .

$$C = \frac{1}{r_0} - \frac{G \cdot M}{h^2}$$

... Eq. (27)

Knowing the constant of the trajectory, the eccentricity can be found.

$$\varepsilon = \frac{C \cdot h^2}{G \cdot M} = \frac{h^2}{G \cdot M} \cdot \left(\frac{1}{r_0} - \frac{G \cdot M}{h^2} \right)$$

... Eq. (28)

Which simplifies to

$$\varepsilon = \frac{h^2}{r_0 \cdot G \cdot M} - 1 = \frac{r_0 \cdot v_0^2}{G \cdot M} - 1$$

... Eq. (29)

The different cases of eccentricity can approximate key velocities from eq. (29). To launch a rocket into a circular orbit of radius r_0 , it must reach a velocity where $\varepsilon = 0$.

$$\varepsilon = 0$$

$$v_0 = \sqrt{\frac{G \cdot M}{r_0}} = v_{circ}$$

... Eq. (30)

An elliptical path occurs as the eccentricity is between 0 and 1.

$$0 < \epsilon < 1$$

$$v_{circ} < v_0 < \sqrt{\frac{2 \cdot G \cdot M}{r_0}}$$

... Eq. (31)

The minimum velocity required to escape an orbit happens when the eccentricity is 1 and the object is travelling in a parabola.

$$\epsilon = 1$$

$$v_0 = \sqrt{\frac{2 \cdot G \cdot M}{r_0}} = v_{esc}$$

... Eq. (32)

Finally, a hyperbolic path requires a velocity that puts the object in an eccentricity greater than 1.

$$\epsilon > 1$$

$$v_0 > \sqrt{\frac{2 \cdot G \cdot M}{r_0}}$$

... Eq. (33)

▼ Properties of an Ellipse

An elliptical orbit contains two foci, denoted as $F1$ and $F2$ in the below figure. The radius from the center of the ellipse to the farthest edge is the **semi major axis**, a . The distance from the center to the closest edge of the ellipse is the **semi minor axis**, b .

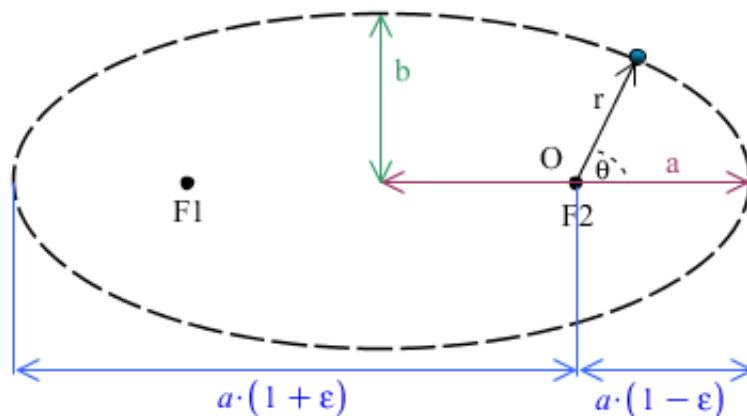


Figure 4: An object in an elliptical orbit

A planet or a large mass lies at a focus O of the ellipse. The eccentricity of this elliptical orbit can be described with the largest and smallest r values.

$$\epsilon = \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}}$$

... Eq. (34)

These radial values can be calculated using eq. (25) at $r_{\min} = r(\theta=0)$ and $r_{\max} = r(\theta=\pi)$. These values help calculate key characteristics of the ellipse such as the magnitude of the semi major and minor axes, a and b .

$$a = \frac{1}{2} \cdot (r_{\max} + r_{\min}) \quad \text{and} \quad b = \sqrt{r_{\max} \cdot r_{\min}}$$

... Eq. (35a, 35b)

The period of an object in an elliptical orbit is given by

$$T = \frac{2 \cdot \pi \cdot a \cdot b}{h}$$

... Eq. (36)

Kepler's Laws of Planetary Motion

The preceding equations assumed that the secondary mass such as a satellite or rocket is negligible in the calculations. However, considering the moon moving about the earth, the second mass is a rather large factor in determining its orbit. The approximations obtained from these equations aren't entirely accurate because of the influence of the moon on

Earth's orbit. This problem also arises when estimating the path of the planets, however these equations do provide a reasonably good approximation.

Johann Kepler, a German astronomer, developed his 3 laws which govern the motion of the planets.

1. Each planet describes an elliptical orbit with the sun at one of its two foci.
2. The radius vector drawn from the sun to a planet sweeps equal areas in equal times.
3. The square of the orbital period of a planet is proportional to the cube of the semi-major axis of their orbit.

Proof of Kepler's Third Law

Consider a planet moving around the sun. Using eq. (25), the position at $r(\theta = 0)$ and $r(\theta = \pi)$ are given by

$$\frac{1}{r_{\min}} = \frac{G \cdot M}{h^2} \cdot (1 + \epsilon)$$

$$\frac{1}{r_{\max}} = \frac{G \cdot M}{h^2} \cdot (1 - \epsilon)$$

... Eq. (37)

Adding the equations cancels the eccentricity. Simplify and isolate for the angular momentum.

$$h^2 = 2 \cdot G \cdot M \cdot \frac{r_{\min} \cdot r_{\max}}{r_{\min} + r_{\max}}$$

... Eq. (38)

Recall the semi major and semi minor axes from eq. (35). Using these in eq. (38) reduces the form.

$$h^2 = G \cdot M \cdot \frac{b^2}{a}$$

... Eq. (39)

Kepler's third law relates to the period of an orbit. Using the squared form of the period eq. (36), and the angular momentum in eq. (39), the equation reduces to the third law.

$$T^2 = 4 \cdot \pi^2 \cdot a^2 \cdot b^2 \cdot \left(\frac{a}{b^2} \cdot \frac{1}{G \cdot M} \right) = \frac{4 \cdot \pi^2}{G \cdot M} \cdot a^3$$

Examples with MapleSim

Example 1: Maximum Altitude

Problem Statement: A satellite enters an orbit 500km above the earth. It's velocity at this point is 36 900 km/h perpendicular to the earth. What will the maximum altitude that the satellite will reach?

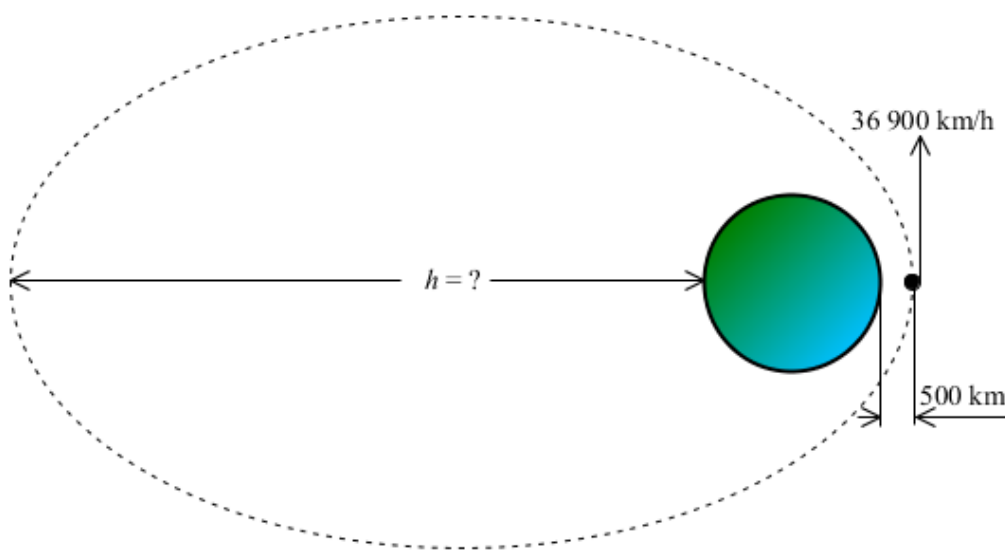


Figure 5: A satellite in an elliptic orbit about Earth

Analytical Solution

restart;

Data:

Convert units to SI units.

$R := 6371E+3 :$

$M := 5.972E+24 :$

$G := 6.67384E-11 :$

$$r_0 := 500E+3 :$$

$$v_0 := \frac{36900}{3.6} : \text{Converted to m/s}$$

Solution:

This satellite has entered an orbit, so its motion can be estimated using eq. (23).

$$eq1 := \frac{1}{r} = \frac{G \cdot M}{h^2} + C \cdot \cos(\theta) :$$

The initial height of the satellite must include the radius of the earth.

$$r_0 := r_0 + R :$$

The angular acceleration will remain constant throughout the orbit. Use the supplied initial velocity and radius at $\theta = 0$.

$$h := r_0 \cdot v_0 :$$

Knowing the initial radius at $\theta = 0$ allows eq.(27) to be used to find the C constant.

$$C := \frac{1}{r_0} - \frac{G \cdot M}{h^2} :$$

All constants have been found. Calculate the maximum radius at $\theta = \pi$.

$$\theta := \text{Pi} :$$

$$sol := \text{solve}(eq1, r) = 6.592415065 \cdot 10^7$$

Subtracting the radius of the earth reveals the maximum altitude of the satellite above earth, in meters.

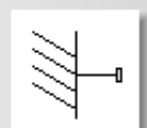


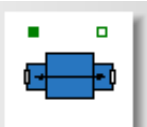

$$sol := \text{abs}(sol) - R = 5.955315065 \cdot 10^7$$


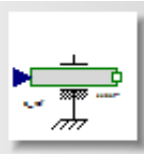
Therefore, the satellite reaches a maximum height of 59 553 km.

MapleSim Solution

Step 1: Insert Components

Drag the following components into a new workspace.

Component	Location
 Fixed Frame	Multibody > Bodies and Frames
 Rigid Body	Multibody > Bodies and Frames
 Revolute	Multibody > Joints and Motions
 Prismatic	Multibody > Joints and Motions
 Constant Speed	1D Mechanical > Rotational > Motion Drivers

 <p>Angle Sensor</p>	<p>1D Mechanical > Rotational > Sensors</p>
 <p>Translational Position</p>	<p>1D Mechanical > Translational > Motion Drivers</p>

Step 2: Connect the components.

Connect the components as shown in the diagram below.

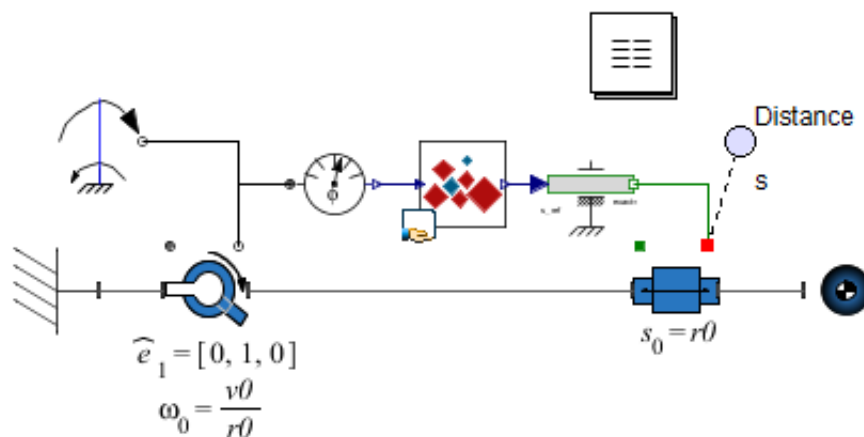



Figure 6: Model Diagram

Step 3: Add Parameters

1. Add a parameters block to the model. Select the parameter icon  and click on the workspace. Open the parameters block and enter the following values: $r\theta = 500E+3$; $v\theta = 36900/3.6$; $R = 6371E+3$.

Name	Type	Default Value	Default Units	Description
r_0	Real	$500 \cdot 10^3 + R$		m
v_0	Real	$36900 \cdot \frac{1}{3.6}$		m/s
R	Real	$6371 \cdot 10^3$		m

Figure 7: Parameter values for MapleSim model

Step 4: Enter Block Values

1. Select the **Revolute** block. On the 'Inspector' pane, change the axis of rotation \widehat{e}_1 parameter to $[0,1,0]$. Change the initial conditions $IC_{\theta, \omega}$ to 'Treat as Guess', and set the speed of rotation ω_0 to $\frac{v_0}{r_0}$.
2. For the **Prismatic** block, change $IC_{s,v}$ to 'Treat as Guess' and set the initial displacement s_0 to r_0 .

Step 5: Create the Custom Component.

1. Add a new custom component to the MapleSim document. From the 'View' menu, select 'Create Attachment'. Select 'Custom Component' and create the attachment.
2. Enter the governing motion equation (eq. 23) at 'eq', denoting θ as $ang(t)$ and r as $r(t)$ to specify these as inputs and outputs. Press enter/return to save the equation.
3. Enter the constant values under 'params'. Angular momentum h will be the product of the initial radius and velocity. Enter the constant C from eq. (27). Press enter/return.

$$\text{eq} := \left[\frac{1}{r(t)} = \frac{GM}{h^2} + C \cdot \cos(\text{ang}(t)) \right]$$

$$\left[\frac{1}{r(t)} = \frac{GM}{h^2} + C \cos(\text{ang}(t)) \right] \quad (3.1)$$

$$\text{params} := \left[G = 6.67384 \cdot 10^{-11}, M = 5.972 \cdot 10^{24}, h = r0 \cdot v0, C = \frac{1}{r0} - \frac{GM}{h^2} \right]$$

$$\left[G = 6.673840000 \cdot 10^{-11}, M = 5.972000000 \cdot 10^{24}, h = r0 \cdot v0, C = \frac{1}{r0} - \frac{GM}{h^2} \right] \quad (3.2)$$

$$\text{initialconditions} := []$$

$$[] \quad (3.3)$$

Figure 8: Equations for custom component

4. Under the **Component Ports** section, click 'Clear All Ports' and add 2 new ports. For the first port, set the port type to 'Signal Input' and on the drop down menu for the value, select $\text{ang}(t)$.
5. For the second port, set the port type to 'Signal Output' and select the $r(t)$ value.
6. Generate the MapleSim component. Connect it to the model appropriately. The block icon can be found on the 'Project' tab under 'Definitions > Components'.

Step 6: Connect probes

1. Connect a probe to flange_b of the **Prismatic** block to measure the distance from the center of the earth to the satellite.

Step 3: Run Simulation

Run the simulation and probe the generated graph. Subtract the radius of the earth from the generated value to obtain the maximum altitude of the satellite.

▼ Results

The simulation generates the following plot.

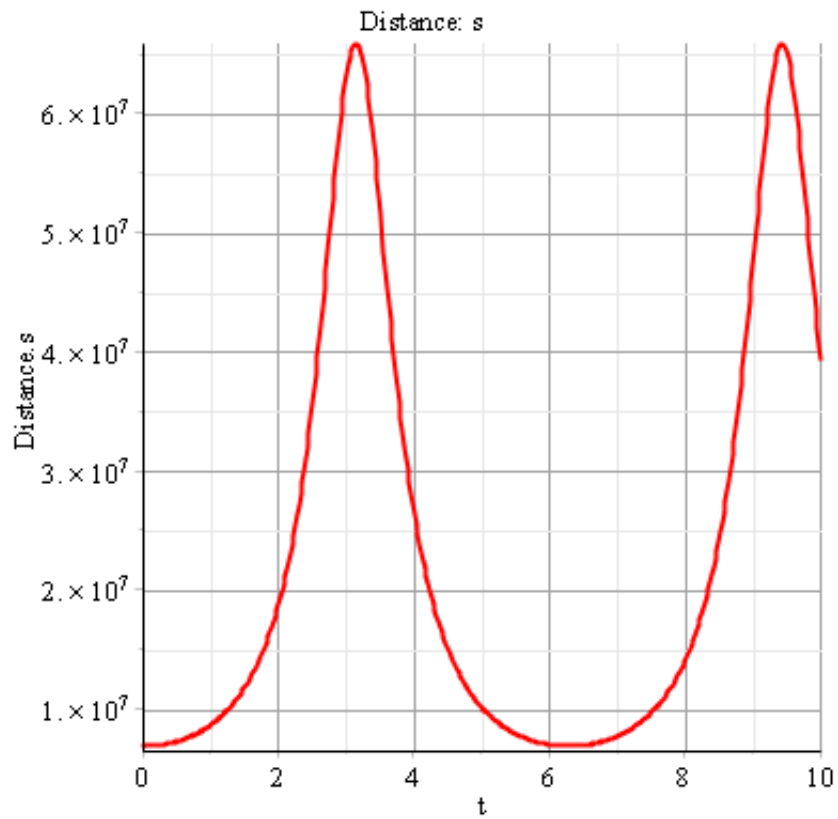


Figure 9: Simulation results

The maximum distance measured from the plot is $6.592 \cdot 10^7$ m. Subtract the radius of the earth, 6371 km, from this value.

$$6.592E+7 - 6371E+3 = 5.9549 \cdot 10^7 \text{ meters}$$

References:

Beer et al. "Vector Mechanics for Engineers: Dynamics", 10th Edition. 1221 Avenue of the Americas, New York, NY, 2013, McGraw-Hill, Inc.