Why I needed Maple to make cream cheese frosting

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Today I encountered the first kitchen math problem I couldn't solve on scrap paper. A recipe for cream cheese frosting called for 8 oz. (about 240 grams) of cream cheese. Unfortunately, I didn't have a kitchen scale, and the product I bought came in a 400 gram tub, shown here.

How deep into the tub should I scoop to get 240 grams? The difficulty lies in the shape of the container.

If the tub were a simple cylinder, I'd just scoop out $\frac{240\ grams}{400\ grams} = \frac{3}{5}$ of the container, which in the case of a cylinder is easy to gauge with the eye or a ruler.

If the tub were a cone, I'd need a calculator with a cube-root button (the formula is easy to derive) but nothing fancier.

However, this tub of cream cheese is no ordinary cone -- it's a *truncated* cone, whose volume formula is the more cumbersome

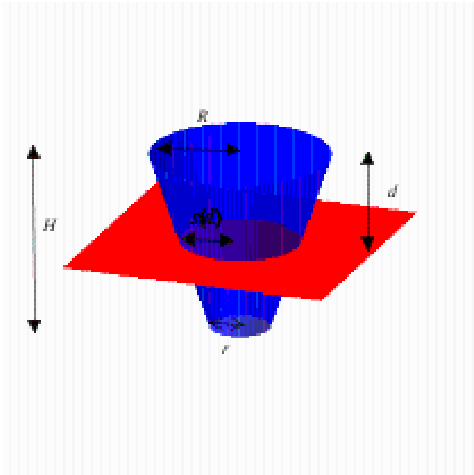
$$V = \frac{\pi H \left(R^2 + R\, r + r^2\right)}{3}$$

(I love how this formula reduces to the volume of a cylinder when $R = r$!)

It took about a page of scratch work to see that finishing this problem would involve solving a rather messy cubic equation. That's when I brought in Maple.

[By the way, while I was bent over the computer doing all this, my 8-year-old took over the frosting making, made an eye-ball guess at the amount of cream cheese to scoop out, and produced a frosting that tasted wonderful. Though preempted, I took great satisfaction in having derived the correct scooping depth to the tenth of a millimeter, just in time to see the finished, frosted cake being served.]

The approach



Let *d* = the depth of the tub we scoop out. This is the quantity we wish to determine.

Let $s(d)$ = the radius of the tub at depth *d*.

Let $f(d)$ = the fraction of the tub's total volume that's removed if we scoop to depth *d*, with *R*, *r*, and *H* considered constants.

Note that $f(H) = 1$.

Let f_{req} = the fraction of the tub's total volume the recipe requires. In my case,

$$f_{req} = \frac{240\ grams}{400\ grams} = \frac{3}{5}$$

To determine *d*, we

- ☐ Derive a formula for $f(d)$ using similar triangles.
- ☐ Solve the equation $f(d) = f_{req}$ for *d* in terms of the constants *R*, *r*, *H*, and f_{req}.
- ☐ Measure *R*, *r*, *H*, and substitute their values into the solution.

Deriving the cubic equation

The volume of the whole tub is given by the volume formula for truncated cones

$$VolumeWholeTub = \frac{1}{3}\pi H\left(R^2 + R\,r + r^2\right)$$

The volume that gets removed if we scoop to depth *d* is given by formula evaluated at *H* = *d* and *r* = *s*.

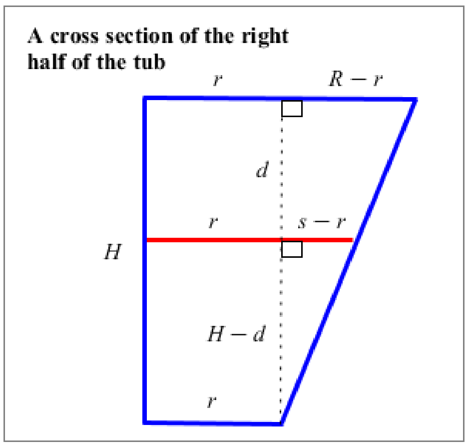
$$ScoopedVolume = \frac{1}{3}\,\pi\,d\,\left(R^2 + R\,s + s^2\right)$$

The fraction of the tub's total volume that's been scooped out as a function of *d* is the ratio of these two formulae. (The bothersome $\frac{\pi}{3}$ nicely cancels out.)

$$f(d) = \frac{rhs(\mathbf{(2.2)})}{rhs(\mathbf{(2.1)})}$$

$$f(d) = \frac{d\,\left(R^2 + R\,s + s^2\right)}{H\,\left(R^2 + R\,r + r^2\right)}$$

The key then is to express *s* in terms of *d*, *R*, *r*, and *H*.



By similar triangles, we have

$$\frac{s - r}{R - r} = \frac{H - d}{H}$$

isolate for s →

$$s = \frac{(H - d)\,(R - r)}{H} + r$$

simplify =

$$s = \frac{H\,R - d\,R + d\,r}{H}$$

which we rewrite for the sake of aesthetics as

$$s = R - \frac{(R - r)\,d}{H}$$

Substituting this expression for *s* into , we get

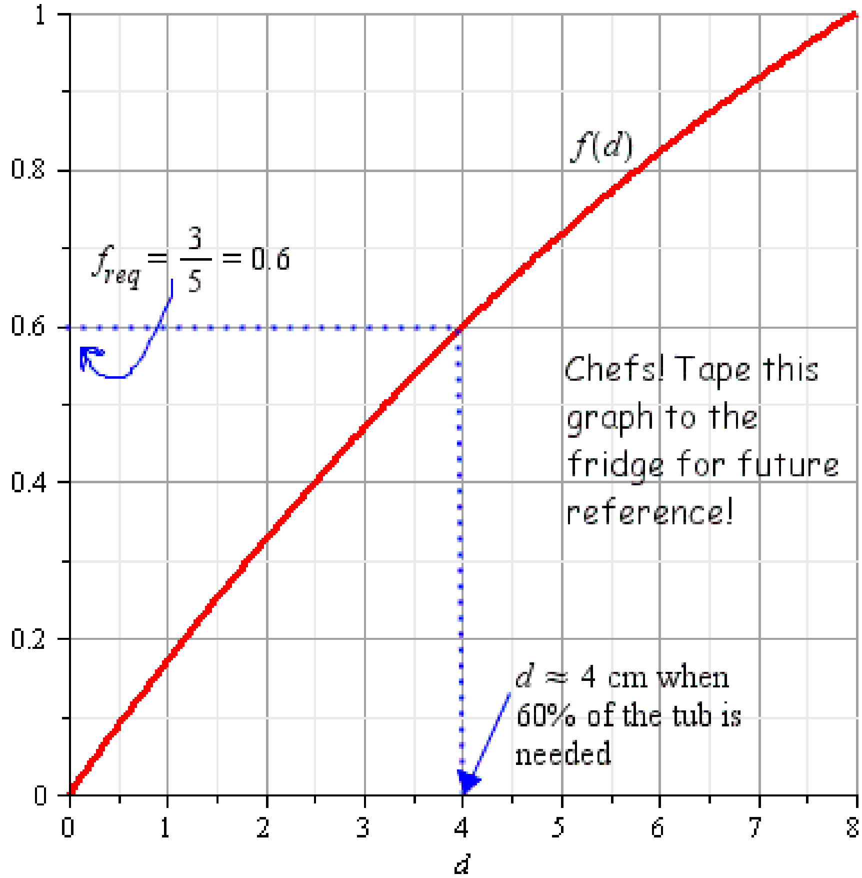
$$f(d) = \frac{1}{H\left(R^2 + R\,r + r^2\right)}\left(d\left(R^2 + R\left(R - \frac{(R-r)\,d}{H}\right) + \left(R - \frac{(R-r)\,d}{H}\right)^2\right)\right)$$

which is cubic in *d*. We can see this by expanding and collecting the RHS of w.r.t. *d*

$$f(d) = \frac{(R-r)^2\,d^3}{H^3\left(R^2 + R\,r + r^2\right)} - \frac{3\,R\,(R-r)\,d^2}{H^2\left(R^2 + R\,r + r^2\right)} + \frac{3\,R^2\,d}{H\left(R^2 + R\,r + r^2\right)}$$

I measured my tub of cream cheese with a ruler and got ***R* = 4.5 cm, *r* = 3.0 cm**, and ***H* = 8.0 cm**. With these values, $f(d)$ becomes the following, which is graphed below.

$$f(d) = 0.0001027960526\,d^3 - 0.007401315789\,d^2 + 0.1776315790\,d$$



From the graph, we see that **we should scoop 4 cm of cream cheese from the tub when 60% of the tub's volume is needed**, i.e. $d \approx 4$ cm when $f_{req} = \frac{3}{5} = 0.6$

Finding *d* analytically

Let's verify this result analytically.

Since we're not interested in the complex solutions of cubic equation , we'll tell Maple to work within the domain of real numbers using the following command.

$$with(RealDomain):$$

Now we solve the following equation for *d* and evaluate the solution using the measurements *R* = 4.5 cm, *r* = 3 cm, and *H* = 8 cm

$$\frac{(R-r)^2 d^3}{H^3 (R^2 + R r + r^2)} - \frac{3 R (R-r) d^2}{H^2 (R^2 + R r + r^2)} + \frac{3 R^2 d}{H (R^2 + R r + r^2)} = f_{req} \xrightarrow{\text{solutions for d}}$$

$$\frac{H \left(R + \left(-R^3 + R^3 f_{req} - f_{req} r^3\right)^{1/3}\right)}{R - r} \xrightarrow{\text{evaluate at point}} 4.010672363$$

Wah-la. The result is confirmed. Next time I'll know exactly what to do before my kid arrives on the scene. All chefs should memorize this formula.

Other ways to solve the problem

My wife suggested two embarrassingly simple alternative solutions (naturally after I'd finished doing all of the above):

1. Empty the entire tub of cream cheese onto a cutting board, then reshape it with your hands into a rectangular block, out of which it's easy to cut the required proportion.

2. Go next door and borrow the neighbor's kitchen scale.

Well, sure, if you want to do it the *boring* way. =) Happy cooking!